Equivalence principle in classical electrodynamics

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Short title: **Equivalence principle in electrodynamics**

Produced: → $\boxed{\text{February 5, 2008}}$

http://www.arXiv.org e-Print archive No.: gr-qc/0303002

Subject Classes:
General relativity, Classical electrodynamics

2001 MSC numbers:	***2001 PACS numbers:***
53B05, 53Z05, 78A99	*02.40.Ma, 04.90.+e, 11.15.Kc, 41.90.+e*

Key-Words:
Normal frames, Normal coordinates
Linear connections, Equivalence principle
Classical electrodynamics, Electromagnetic field
Electromagnetic potentials

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Abstract

The principle of equivalence in gravitational physics and its mathematical base are reviewed. It is demonstrated how this principle can be realized in classical electrodynamis. In general, it is valid at any given single point or along a path without selfintersections unless the field considered satisfies some conditions.

1. Introduction

The equivalence principle is a well known statement in gravitational physics [1, 2, 3, 4, 5, 6, 7]. The range of its validity was extended in [8] to cover classical gauge theories. The present paper concentrate to its realization in classical electrodynamics.

Section 2 introduces the rigorous mathematical background on which the equivalence principle is based. Sections 2.1 and 2.2 contain a brief review of the concepts of linear connection and linear transport along paths in vector bundles. Some links between these objects are investigated in Sect. 2.3. In Sect. 2.4 are considered the so-called normal frames for linear connections and linear transports along paths in vector bundles. The importance of these concepts for the physics comes from the fact that they turn to be the mathematical equivalent to the physical notion of an inertial frame of reference.

In 3 are reviewed some properties of the electromagnetic potentials in classical electrodynamics. Special attention is paid to their geometrical interpretation as (3-index) coefficients of a linear connection (parallel transport along paths) in one-dimensional vector bundle. The equivalence principle in gravity physics is considered in Sect. 4. Section 5 is devoted to the principle of equivalence in classical electrodynamics. As in a gravity theory (based on linear connection(s)), it can be formulated as the assertion for coincidence of inertial and normal frames for a given electromagnetic field. Similarly it is identically valid at any single point or injective path in the spacetime and on other subsets it does not hold generally if some additional conditions are fulfilled.

Section 6 ends the work with some concluding remarks.

2. Normal frames for linear transports along paths and linear connections

The general theory of frames normal for a broad class of derivations, in particular covariant derivatives (linear connections), and linear transports along paths (which in particular can be parallel transports generated by linear connections) is developed in [9, 10, 11, 12, 8] and in the references therein. The material in this section is abstracted from these works and concerns mainly linear connections. Since the classical electromagnetic field is naturally described via linear connections (see Sect. 3), this is done with the intention for applying the general theory mentioned to the classical electrodynamics (see Sect. 5).

2.1. Linear connections in vector bundles

Different equivalent definitions of a (linear) connection in vector bundles are known and in current usage [13, 14, 15, 16]. The most suitable one for our purposes is given in [17, p. 223] or [18, p. 281] (see also [14, theorem 2.52]).

Suppose (E, π, M), E and M being finite-dimensional C^∞ manifolds, be C^∞ $\mathbb{K}$-vector bundle [14] with bundle space E, base M, and projection $\pi\colon E \to M$. Here $\mathbb{K}$ stands for the field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex ones. Let $\mathrm{Sec}^k(E, \pi, M)$, $k = 0, 1, 2, \dots$ be the set (in fact the module) of C^k sections of (E, π, M) and $\mathfrak{X}(M)$ the one of vector fields on M.

Definition 2.1. Let $V, W \in \mathfrak{X}(M)$, $\sigma, \tau \in \mathrm{Sec}^1(E, \pi, M)$, and $f\colon M \to \mathbb{K}$ be a C^∞ function. A mapping $\nabla\colon \mathfrak{X}(M) \times \mathrm{Sec}^1(E, \pi, M) \to \mathrm{Sec}^0(E, \pi, M)$, $\nabla\colon (V, \sigma) \mapsto \nabla_V \sigma$, is called a *(linear)*

connection in (E, π, M) if:

$$\nabla_{V+W}\sigma = \nabla_V\sigma + \nabla_W\sigma, \tag{2.1a}$$
$$\nabla_{fV}\sigma = f\nabla_V\sigma, \tag{2.1b}$$
$$\nabla_V(\sigma+\tau) = \nabla_V(\sigma) + \nabla_V(\tau), \tag{2.1c}$$
$$\nabla_V(f\sigma) = V(f)\cdot\sigma + f\cdot\nabla_V(\sigma). \tag{2.1d}$$

Remark 2.1. Rigorously speaking, ∇, as defined by definition 2.1, is a covariant derivative operator in (E, π, M) — see [14, definition 2.51] — but, as a consequence of [14, theorem 2.52], this cannot lead to some ambiguities.

Remark 2.2. Since $V(a) = 0$ for every $a \in \mathbb{K}$ (considered as a constant function $M \to \{a\}$), the mapping $\nabla\colon (V,\sigma) \mapsto \nabla_V\sigma$ is $\mathbb{K}$-linear with respect to both its arguments.

Let $\{e_i : i = 1,\dots,\dim\pi^{-1}(x)\}$, $x \in M$ and $\{E_\mu : \mu = 1,\dots,\dim M\}$ be frames over an open set $U \subseteq M$ in, respectively, (E, π, M) and the tangent bundle $(T(M), \pi_T, M)$ over M, i.e. for every $x \in U$, the set $\{e_i|_x\}$ forms a basis of the fibre $\pi^{-1}(x)$ and $\{E_\mu|_x\}$ is a basis of the space $T_x(M) = \pi_T^{-1}(x)$ tangent to M at x. Let us write $\sigma = \sigma^i e_i$ and $V = V^\mu E_\mu$, where here and henceforth the Latin (resp. Greek) indices run from 1 to the dimension of (E, π, M) (resp. M), the Einstein summation convention is assumed, and $\sigma^i, V^\mu\colon U \to \mathbb{K}$ are some C^1 functions. Then, from definition 2.1, one gets

$$\nabla_V\sigma = V^\mu\big(E_\mu(\sigma^i) + \Gamma^i{}_{j\mu}\sigma^j\big)e_i \tag{2.2}$$

where $\Gamma^i{}_{j\mu}\colon U \to \mathbb{K}$, called *coefficients* of ∇, are given by

$$\nabla_{E_\mu}e_j =: \Gamma^i{}_{j\mu}e_i. \tag{2.3}$$

Evidently, by virtue of (2.2), the knowledge of $\{\Gamma^i{}_{j\mu}\}$ in a pair of frames $(\{e_i\},\{E_\mu\})$ over U is equivalent to the one of ∇ as any transformation $(\{e_i\},\{E_\mu\}) \mapsto (\{e'_i = A^j_i e_j\}, \{E'_\mu = B^\nu_\mu E_\nu\})$ with non-degenerate matrix-valued functions $A = [A^j_i]$ and $B = [B^\nu_\mu]$ on U implies $\Gamma^i{}_{j\mu} \mapsto \Gamma'^i{}_{j\mu}$ with

$$\Gamma'^i{}_{j\mu} = \sum_{\nu=1}^{\dim M}\sum_{k,l=1}^{\dim\pi^{-1}(x)} B^\nu_\mu (A^{-1})^i_k A^l_j \Gamma^k{}_{l\nu} + \sum_{\nu=1}^{\dim M}\sum_{k=1}^{\dim\pi^{-1}(x)} B^\nu_\mu (A^{-1})^i_k E_\nu(A^k_j). \tag{2.4}$$

which in a matrix form reads

$$\Gamma'_\mu = B^\nu_\mu A^{-1}\Gamma_\nu A + A^{-1}E'_\mu(A) = B^\nu_\mu A^{-1}\big(\Gamma_\nu A + E_\nu(A)\big) \tag{2.5}$$

where $\Gamma_\mu := [\Gamma^i{}_{j\mu}]_{i,j=1}^{\dim\pi^{-1}(x)}$, $x \in M$, and $\Gamma'_\mu := [\Gamma'^i{}_{j\mu}]_{i,j=1}^{\dim\pi^{-1}(x)}$.

2.2. Linear transports along paths in vector bundles

To begin with, we recall some definitions and results from the paper [12].[1] Below we denote by $\mathrm{PLift}^k(E, \pi, M)$ the set of liftings of C^k paths from M to E such that the lifted paths are of class C^k, $k = 0, 1, \dots$. Let $\gamma\colon J \to M$, J being real interval, be a path in M.

Definition 2.2. A *linear transport along paths in vector bundle* (E, π, M) is a mapping L assigning to every path γ a mapping L^γ, *transport along* γ, such that $L^\gamma\colon (s,t) \mapsto L^\gamma_{s\to t}$ where the mapping

$$L^\gamma_{s\to t}\colon \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t)) \qquad s,t \in J, \tag{2.6}$$

[1] In [12] is assumed $\mathbb{K} = \mathbb{C}$ but this choice is insignificant.

called *transport along* γ *from* s *to* t, has the properties:

$$L^{\gamma}_{s\to t}\circ L^{\gamma}_{r\to s}=L^{\gamma}_{r\to t}, \qquad r,s,t\in J, \tag{2.7}$$
$$L^{\gamma}_{s\to s}=\mathsf{id}_{\pi^{-1}(\gamma(s))}, \qquad s\in J, \tag{2.8}$$
$$L^{\gamma}_{s\to t}(\lambda u+\mu v)=\lambda L^{\gamma}_{s\to t}u+\mu L^{\gamma}_{s\to t}v, \qquad \lambda,\mu\in\mathbb{K}, \quad u,v\in\pi^{-1}(\gamma(s)), \tag{2.9}$$

where $\circ$ denotes composition of maps and id_X is the identity map of a set X.

Definition 2.3. A *derivation along paths in* (E,π,M) or a *derivation of liftings of paths in* (E,π,M) is a mapping

$$D\colon\ \mathrm{PLift}^1(E,\pi,M)\to\mathrm{PLift}^0(E,\pi,M) \tag{2.10a}$$

which is $\mathbb{K}$-linear,

$$D(a\lambda+b\mu)=aD(\lambda)+bD(\mu) \tag{2.11a}$$

for $a,b\in\mathbb{K}$ and $\lambda,\mu\in\mathrm{PLift}^1(E,\pi,M)$, and the mapping

$$D^{\gamma}_{s}\colon\ \mathrm{PLift}^1(E,\pi,M)\to\pi^{-1}(\gamma(s)), \tag{2.10b}$$

defined via $D^{\gamma}_{s}(\lambda):=\big((D(\lambda))(\gamma)\big)(s)=(D\lambda)_{\gamma}(s)$ and called *derivation along* $\gamma\colon J\to M$ *at* $s\in J$, satisfies the 'Leibnitz rule':

$$D^{\gamma}_{s}(f\lambda)=\frac{\mathrm{d}f_{\gamma}(s)}{\mathrm{d}s}\lambda_{\gamma}(s)+f_{\gamma}(s)D^{\gamma}_{s}(\lambda) \tag{2.11b}$$

for every

$$f\in\mathrm{PF}^1(M):=\{\varphi|\varphi\colon\gamma\mapsto\varphi_{\gamma},\ \gamma\colon J\to M,\ \varphi_{\gamma}\colon J\to\mathbb{K}\text{ being of class }C^1\}.$$

The mapping

$$D^{\gamma}\colon\ \mathrm{PLift}^1(E,\pi,M)\to\mathrm{P}\big(\pi^{-1}(\gamma(J))\big):=\{\text{paths in }\pi^{-1}(\gamma(J))\}, \tag{2.10c}$$

defined by $D^{\gamma}(\lambda):=(D(\lambda))|_{\gamma}=(D\lambda)_{\gamma}$, is called *derivation along* γ.

If $\gamma\colon J\to M$ is a path in M and $\{e_i(s;\gamma)\}$ is a basis in $\pi^{-1}(\gamma(s))$,[2] in the frame $\{e_i\}$ over $\gamma(J)$ the *components (matrix elements)* ${L^{i}}_{j}\colon U\to\mathbb{K}$ of a linear transport L along paths and the ones of a derivation D along paths in vector bundle (E,π,M) are defined through, respectively,

$$L^{\gamma}_{s\to t}\big(e_i(s;\gamma)\big)=:{L^{j}}_{i}(t,s;\gamma)e_j(t;\gamma) \qquad s,t\in J, \tag{2.12}$$
$$D^{\gamma}_{s}\hat{e}_j=:{\Gamma^{i}}_{j}(s;\gamma)e_i(s;\gamma) \qquad s\in J \tag{2.13}$$

where $\hat{e}_i\colon\gamma\mapsto e_i(\cdot;\gamma)$ is a lifting of γ generated by e_i.

It is a simple exercise to verify that the components of L and D uniquely define (locally) their action on $u=u^ie_i(s;\gamma)$ and $\lambda\in\mathrm{PLift}^1(E,\pi,M)$, $\lambda\colon\gamma\mapsto\lambda_{\gamma}=\lambda^i_{\gamma}\hat{e}_i$, according to

$$L^{\gamma}_{s\to t}u=:{L^{i}}_{j}(t,s;\gamma)u^je_i(t;\gamma) \tag{2.14}$$
$$D^{\gamma}_{s}\lambda=:\Big(\frac{\mathrm{d}\lambda^i_{\gamma}(s)}{\mathrm{d}s}+{\Gamma^{i}}_{j}(s;\gamma)\lambda^j_{\gamma}(s)\Big)e_i(s;\gamma) \tag{2.15}$$

and that a change $\{e_i(s;\gamma)\}\mapsto\{e'_i(s;\gamma)=A^j_i(s;\gamma)e_j(s;\gamma)\}$, with a non-degenerate matrix-valued function $A(s;\gamma):=[A^j_i(s;\gamma)]$, implies the transformations

$$\boldsymbol{L}(t,s;\gamma):=\big[{L^{j}}_{i}(t,s;\gamma)\big]\mapsto\boldsymbol{L}'(t,s;\gamma)=A^{-1}(t;\gamma)\boldsymbol{L}(t,s;\gamma)A(s;\gamma) \tag{2.16}$$

[2] If there are $s_1,s_2\in J$ such that $\gamma(s_1)=\gamma(s_2):=y$, the vectors $e_i(s_1;\gamma)$ and $e_i(s_2;\gamma)$ need not to coincide. So, if this is the case, the bases $\{e_i(s_1;\gamma)\}$ and $\{e_i(s_2;\gamma)\}$ in $\pi^{-1}(y)$ may turn to be different.

$$\boldsymbol{\Gamma}(s;\gamma) := \left[\Gamma^i{}_j(s;\gamma)\right] \mapsto \boldsymbol{\Gamma}'(s;\gamma) = A^{-1}(s;\gamma)\boldsymbol{\Gamma}(s;\gamma)A(s;\gamma) + A^{-1}(s;\gamma)\frac{\mathrm{d}A(s;\gamma)}{\mathrm{d}s}. \qquad (2.17)$$

A crucial role further will be played by the *coefficients* $\Gamma^i{}_j(s;\gamma)$ in a frame $\{e_i\}$ of linear transport L,

$$\Gamma^i{}_j(s;\gamma) := \frac{\partial L^i{}_j(s,t;\gamma)}{\partial t}\bigg|_{t=s} = -\frac{\partial L^i{}_j(s,t;\gamma)}{\partial s}\bigg|_{t=s}. \qquad (2.18)$$

The usage of the same notation for the *coefficients* of a transport L and *components* of derivation D along paths is not accidental and finds its reason in the following fundamental result [12, sec. 2]. Call a transport L differentiable of class C^k, $k = 0, 1, \dots$ if its matrix $\boldsymbol{L}(t,s;\gamma)$ has C^k dependence on t (and hence on s — see [12, sec. 2]). *Every C^1 linear transport L along paths generates a derivation D along paths via*

$$D^\gamma_s(\lambda) := \lim_{\varepsilon\to 0}\Big\{\frac{1}{\varepsilon}\big[L^\gamma_{s+\varepsilon\to s}\lambda_\gamma(s+\varepsilon) - \lambda_\gamma(s)\big]\Big\} \qquad (2.19)$$

for every lifting $\lambda \in \mathrm{PLift}^1(E,\pi,M)$ with $\lambda\colon \gamma \mapsto \lambda_\gamma$ and conversely, for any derivation D along paths there exists a unique linear transport along paths generating it via (2.19). Besides, *if L and D are connected via* (2.19)*, the coefficients of L coincide with the components of D.* In short, there is a bijective correspondence between linear transports and derivations along paths given locally through the equality of their coefficients and components respectively.

More details and results on the above items can be found in [12].

2.3. Links between linear connections and linear transports

Suppose $\gamma\colon J \to M$ is a C^1 path and $\dot\gamma(s)$, $s \in J$, is the vector tangent to γ at $\gamma(s)$ (more precisely, at s). Let ∇ and D be, respectively, a linear connection and derivation along paths in vector bundle (E,π,M), and in a pair of frames $(\{e_i\},\{E_\mu\})$ over some open set in M the coefficients of ∇ and the components of D be $\Gamma^i{}_{j\mu}$ and $\Gamma^i{}_j$ respectively, i.e. $\nabla_{E_\mu}e_i = \Gamma^j{}_{i\mu}e_j$ and $D^\gamma_s\hat e_i = \Gamma^j{}_i e_j(\gamma(s))$ with $\hat e_i\colon \gamma \mapsto \hat e_i|_\gamma\colon s \mapsto e_i(\gamma(s))$ being lifting of paths generated by e_i. If $\sigma = \sigma^i e_i \in \mathrm{Sec}^1(E,\pi,M)$ and $\hat\sigma \in \mathrm{PLift}(E,\pi,M)$ is given via $\hat\sigma\colon \gamma \mapsto \hat\sigma_\gamma := \sigma\circ\gamma$, then (2.15) implies

$$D^\gamma_s\hat\sigma = \Big(\frac{\mathrm{d}\sigma^i(\gamma(s))}{\mathrm{d}s} + \Gamma^i{}_j(s;\gamma)\sigma^j(\gamma(s))\Big)e_i(\gamma(s))$$

while, if $\gamma(s)$ is not a self-intersection point for γ, equation (2.2) leads to

$$(\nabla_{\dot\gamma}\sigma)|_{\gamma(s)} = \Big(\frac{\mathrm{d}\sigma^i(\gamma(s))}{\mathrm{d}s} + \Gamma^i{}_{j\mu}(\gamma(s))\sigma^j(\gamma(s))\dot\gamma^\mu(s)\Big)e_i(\gamma(s)).$$

Obviously, we have

$$D^\gamma_s\hat\sigma = (\nabla_{\dot\gamma}\sigma)|_{\gamma(s)} \qquad (2.20)$$

for every σ iff

$$\Gamma^i{}_j(s;\gamma) = \Gamma^i{}_{j\mu}(\gamma(s))\dot\gamma^\mu(s) \qquad (2.21)$$

which, in matrix form reads

$$\boldsymbol{\Gamma}(s;\gamma) = \Gamma_\mu(\gamma(s))\dot\gamma^\mu(s). \qquad (2.22)$$

A simple algebraic calculation shows that this equality is invariant under changes of the frames $\{e_i\}$ in (E,π,M) and $\{E_\mu\}$ in $(T(M),\pi_T,M)$. Besides, if (2.21) holds, then $\boldsymbol{\Gamma}$ transforms according to (2.17) iff Γ_μ transforms according to (2.5).

The above considerations are a hint that the linear connections should, and in fact can, be described in terms of derivations or, equivalently, linear transports along paths; the second description being more relevant if one is interested in the parallel transports generated by connections.

Theorem 2.1. *If ∇ is a linear connection, then there exists a derivation D along paths such that* (2.20) *holds for every C^1 path $\gamma\colon J \to M$ and every $s \in J$ for which $\gamma(s)$ is not self-intersection point for γ.*[3] *The matrix of the components of D is given by* (2.22) *for every C^1 path $\gamma\colon J \to M$ and $s \in J$ such that $\gamma(s)$ is not a self-intersection point for γ. Conversely, given a derivation D along path whose matrix along any C^1 path $\gamma\colon J \to M$ has the form* (2.22) *for some matrix-valued functions Γ_μ, there is a unique linear connection ∇ whose matrices of coefficients are exactly Γ_μ and for which, consequently,* (2.20) *is valid at the not self-intersection points of γ.*

Proof. NECESSITY. If Γ_μ are the matrices of the coefficients of ∇ in some pair of frames $(\{e_i\},\{E_\mu\})$, define the matrix $\mathbf{\Gamma}$ of the components of D via (2.22) for any $\gamma\colon J \to M$. SUFFICIENCY. Given D for which the decomposition (2.22) holds in $(\{e_i\},\{E_\mu\})$ for any γ. It is trivial to verity that Γ_μ transform according to (2.5) and, consequently, they are the matrices of the coefficients of a linear connection ∇ for which, evidently, (2.20) holds. $\square$

A trivial consequence of the above theorem is the following important result.

Corollary 2.1. *There is a bijective correspondence between the set of linear connections in a vector bundle and the one of derivations along paths in it whose components' matrices admit (locally) the decomposition* (2.22)*. Locally, along a C^1 path γ and pair of frames $(\{e_i\},\{E_\mu\})$ along it, it is given by* (2.22) *in which $\mathbf{\Gamma}$ and Γ_μ are the matrices of the components of a derivation along paths and of the coefficients of a linear connection, respectively.*

Let us now look on the preceding material from the view-point of linear transports along paths and parallel transports generated by linear connections.

Recall (see, e.g., [14, chapter 2]), a section $\sigma \in \mathrm{Sec}^1(E,\pi,M)$ is *parallel along* C^1 path $\gamma\colon J \to M$ with respect to a linear connection ∇ if $(\nabla_{\dot\gamma}\sigma)|_{\gamma(s)} = 0$, $s \in J$.[4] The *parallel transport along* a C^1 path $\alpha\colon [a,b] \to M$, $a,b \in \mathbb{R}$, $a \le b$, generated by ∇ is a mapping

$$P^\alpha\colon \pi^{-1}(\alpha(a)) \to \pi^{-1}(\alpha(b))$$

such that $P^\alpha(u_0) := u(b)$ for every element $u_0 \in \pi^{-1}(\alpha(a))$ where $u \in \mathrm{Sec}^1(E,\pi,M)|_{\alpha([a,b])}$ is the unique solution of the initial-value problem

$$\nabla_{\dot\alpha} u = 0, \qquad u(a) = u_0. \tag{2.23}$$

The *parallel transport* P generated by (assigned to, corresponding to) a linear connection ∇ is a mapping assigning to any $\alpha\colon [a,b] \to M$ the parallel transport P^α along α generated by ∇.

Let D be the derivation along paths corresponding to ∇ according to corollary 2.1. Then (2.20) holds for $\gamma = \alpha$, so (2.23) is tantamount to

$$D^\alpha_s \hat u = 0 \qquad u(a) = u_0 \tag{2.24}$$

where $\hat u\colon \alpha \mapsto \bar u \circ \alpha$ with $\bar u \in \mathrm{Sec}^1(E,\pi,M)$ such that $\bar u|_{\alpha([a,b])} = u$. From here and the results of [12, sec. 2] immediately follows that the lifting $\hat u$ is generated by the unique linear transport P along paths corresponding to D,

$$\hat u\colon \alpha \mapsto \hat u_\alpha := \bar{\mathsf{P}}^\alpha_{a,u_0}, \quad \bar{\mathsf{P}}^\alpha_{a,u_0}\colon s \mapsto \bar{\mathsf{P}}^\alpha_{a,u_0}(s) := \mathsf{P}^\alpha_{a\to s} u_0, \qquad s \in [a,b]. \tag{2.25}$$

[3] In particular, γ can be injective and s arbitrary. If we restrict the considerations to injective paths, the derivation D is unique. The essential point here is that at the self-intersection points of γ, if any, the mapping $\dot\gamma\colon \gamma(s) \mapsto \dot\gamma(s)$ is generally multiple-valued and, consequently, it is not a vector field (along γ); as a result $(\nabla_{\dot\gamma}\sigma)|_{\gamma(s)}$ at them becomes also multiple-valued.

[4] If γ is not injective, here and henceforth $(\nabla_{\dot\gamma}\sigma)|_{\gamma(s)}$ should be replaced by $D^\gamma_s\hat\sigma$, $\hat\sigma\colon \gamma \mapsto \sigma\circ\gamma$, where D is the derivation along paths corresponding to ∇ via corollary 2.1.

Therefore $P^\alpha(u_o) := u(b) = \bar{u}(\alpha(b)) = \hat{u}_\alpha(b) = \mathsf{P}^\alpha_{a\to b}u_0$. Since this is valid for all $u_0 \in \pi^{-1}(\alpha(a))$, we have

$$P^\alpha = \mathsf{P}^\alpha_{a\to b}. \tag{2.26}$$

Theorem 2.2. *The parallel transport P generated by a linear connection ∇ in a vector bundle coincides, in a sense of* (2.26)*, with the unique linear transport P along paths in this bundle corresponding to the derivation D along paths defined, via corollary 2.1, by the connection. Conversely, if P is a linear transport along paths whose coefficients' matrix admits the representation* (2.22)*, then for every $s,t \in [a,b]$*

$$\mathsf{P}^\alpha_{s\to t} = \begin{cases} P^{\alpha|[s,t]} & \text{for } s \le t \\ \left(P^{\alpha|[t,s]}\right)^{-1} & \text{for } s \ge t \end{cases}, \tag{2.27}$$

where P is the parallel transport along paths generated by the unique linear connection ∇ corresponding to the derivation D along paths defined by P.

Proof. The first part of the assertion was proved above while deriving (2.26). The second part is simply the inversion of all logical links in the first one, in particular (2.27) is the solution of (2.26) with respect to P. □

The transport P along paths corresponding according to theorem 2.2 to a parallel transport P or a linear connection ∇ will be called *parallel transport along paths.*

Corollary 2.2. *The local coefficients' matrix Γ of a parallel transport along paths and the coefficients' matrices Γ_μ of the generating it (or generated by it) linear connection are connected via* (2.22) *for every C^1 path $\gamma\colon J \to M$.*

Proof. See theorem 2.2. □

If the coefficients of a linear transport along paths admit a representation (2.22) for any $\gamma\colon J \to U \subseteq M$, we shall call $\Gamma^i{}_{j\mu}\colon U \to \mathbb{K}$ its 3-*index coefficients*, $\Gamma = [\Gamma^i{}_{j\mu}]$ its *coefficient matrices*, and say that it admits 3-index coefficients.

As there is a bijective correspondence between linear transports and derivation along paths (locally given via the coincidence of their respective coefficients and components — see [12]), from corollary 2.2 we get the following result.

Corollary 2.3. *A linear transport along paths admits 3-index coefficients on an* open *set $U \subseteq M$ if and only if it is a parallel transport along paths.*

Notice, if $U \subset M$ is *not* an open set in M, e.g. if it is a submanifold of dimension less than $\dim M$, than corollary 2.3 is generally not valid. The reason for that conclusion is in that, if a transport admits 3-index coefficients $\Gamma^i{}_{j\mu}$ on U, then $\Gamma^i{}_{j\mu} + G^i{}_{j\mu}$ are also its 3-index coefficients for any $G^i{}_{j\mu}\colon U \to \mathbb{K}$ such that $G^i{}_{j\mu}(\gamma(s))\dot\gamma^\mu(s) = 0$ for any C^1 path

$\gamma\colon J \to U$. Consequently, we can assert that $G^i{}_{j\mu} = 0$ if $\dot\gamma(s)$ is an arbitrary vector in $T_{\gamma(s)}(M)$, which is the case when U is an open set in M; if U is a submanifold and $\dim U < \dim M$, then $G^i{}_{j\mu}V^\mu = 0$ with $V_x \in T_x(U)$ does not imply $G^i{}_{j\mu} = 0$ for all $\mu = 1,\dots,\dim M$. So, generally the 3-index coefficients of a linear transport along paths, if any, are not defined uniquely, contrary to the case of parallel transports along paths.

2.4. Normal and strong normal frames

Freely speaking, a normal frame for a derivation (e.g. linear connection) or transport along paths (e.g. parallel one) is a (local) frame in the bundle space in which it looks (locally) as if we are dealing with an ordinary derivation or parallel transport, respectively, in Euclidean space, i.e. in which it looks (locally) Euclidean. That intuitive understanding is formalized in the following definitions in which we restrict ourselves to linear connections due to further considerations in the present work.

Definition 2.4. Given a linear connection ∇ in a vector bundle (E, π, M) and a subset $U \subseteq M$. A *frame* $\{e_i\}$ in E defined over an open subset V of M containing U or equal to it, $V \supseteq U$, is called *normal for* ∇ *over* U if in it and some (and hence any) frame $\{E_\mu\}$ in $T(M)$ over V the coefficients of ∇ vanish everywhere on U. Respectively, $\{e_i\}$ is *normal for* ∇ *along a mapping* $g\colon Q \to M$, $Q \neq \varnothing$, if $\{e_i\}$ is normal for ∇ over $g(Q)$.

Definition 2.5. Given a linear transport L (resp. derivation D) along paths in a vector bundle (E, π, M) and a subset $U \subseteq M$. A frame $\{e_i\}$ in E defined on an open set V containing U, $V \supseteq U$, is called *normal* for L (resp. D) on U if in it vanish the coefficients of L (resp. components of D) along every path $\gamma\colon J \to U$. A frame is called normal (along a path $\gamma\colon J \to M$) for L or D if it is normal for it on $U = M$ (resp. $U = \gamma(J)$).

A linear connection or transport/derivation along paths is called *Euclidean* on $U \subseteq M$ if it admits a frame normal for it on U.

A necessary condition for a linear transport along paths to be Euclidean is provided by the following result [12, proposition 5.1].

Proposition 2.1. *For every Euclidean on $U \subseteq M$ (resp. along a C^1 path $\gamma\colon J \to M$) linear transport L along paths in (E, π, M), E and M being C^1 manifolds, the matrix $\mathbf{\Gamma}$ of its coefficients has the representation*

$$\mathbf{\Gamma}(s; \gamma) = \sum_{\mu=1}^{\dim M} \Gamma_\mu(\gamma(s))\dot{\gamma}^\mu(s) \equiv \Gamma_\mu(\gamma(s))\dot{\gamma}^\mu(s) \tag{2.28}$$

in any frame $\{e_i\}$ along every (resp. the given) C^1 path $\gamma\colon J \to U$, where $\Gamma_\mu = \left[\Gamma^i{}_{j\mu}\right]_{i,j=1}^{\dim \pi^{-1}(x)}$ are some matrix-valued functions, defined on an open set V containing U (resp. $\gamma(J)$') or equal to it, and $\dot{\gamma}^\mu$ are the components of $\dot{\gamma}$ in some frame $\{E_\mu\}$ along γ in the bundle space $T(M)$ tangent to M, $\dot{\gamma} = \dot{\gamma}^\mu E_\mu$.

Combining this result with corollary 2.2, we see that the parallel transports along paths *may* admit normal frames. However, the existence of such frames depends on the subset U on which they are normal. In particular, normal frames always exist if U is a single point and for $U = \gamma(J)$ for some path $\gamma\colon J \to M$; besides, the normal frames are generally anholonomic. For instance, a linear transport L is Euclidean on $U \subseteq M$ iff it is path-independent in U, i.e. iff $L^\gamma_{s\to t}$ depends only on the points $\gamma(s)$ and $\gamma(t)$ but not on the particular path in U connecting them, or iff its matrix in a frame $\{e_i\}$ in E is of the form $\boldsymbol{L}(t, s) = \boldsymbol{F}_0^{-1}(\gamma(t))\boldsymbol{F}_0(\gamma(s))$ for $\gamma\colon J \to U$ and some non-degenerate matrix-valued function $\boldsymbol{F}_0$ on U, or iff its coefficient's matrix in the same frame is $\Gamma(s; \gamma) = \boldsymbol{F}_0^{-1}(\gamma(s))\frac{\mathrm{d}\boldsymbol{F}_0(\gamma(s))}{\mathrm{d}s}$. For details concerning existence, uniqueness and holonomicity of frames normal for linear transports, the reader is referred to [12].

Since in this paper we shall be interested mainly in linear connections, the next considerations will be restricted to frames normal for linear connections and parallel transports along paths (generated by them).

Let ∇ and P be, respectively, a linear connection on M and the parallel transport along paths in (E, π, M) generated by ∇ (see (2.21)). Suppose ∇ and P admit frames normal on a set $U \subseteq M$. Here a natural question arises: what are the links between both types of normal frames, the ones normal for ∇ on U and the ones for P on U?

Recall, if $\Gamma^i{}_{jk}$ are the components of ∇ in a frame $\{E_i\}$, the frame $\{E_i\}$ is normal on $U \subseteq M$ for ∇ or P iff respectively

$$\Gamma^i{}_{jk}(p) = 0 \tag{2.29}$$

$$\Gamma^i{}_j(s; \gamma) = \Gamma^i{}_{jk}(\gamma(s))\dot{\gamma}^k(s) = 0 \tag{2.30}$$

for every $p \in U$, $\gamma\colon J \to U$, and $s \in J$. From these equalities two simple but quite important conclusions can be made: (i) The frames normal for ∇ are normal for P, the opposite being generally not valid, and (ii) in a frame normal for ∇ vanish the 2-index as well as the 3-index coefficients of P.

Definition 2.6. Let P be a parallel transport in (E, π, M) and $U \subseteq M$. A *frame* $\{E_i\}$, defined on an open set containing U, is called *strong normal on* U *for* P if the 3-index coefficients of P in $\{E_i\}$ vanish on U. Respectively, $\{E_i\}$ is *strong normal along* $g\colon Q \to M$ if it is strong normal on $g(Q)$.

Obviously, the set of frames strong normal on U for a parallel transport P coincides with the set of frames normal for the linear connection ∇ generating P.

The above considerations can be generalized directly to linear transports for which 3-index coefficients exist and are fixed (see [12, sec. 7]).

As a sufficient criterion for existence of (strong) normal frames for (a parallel transport generated by) linear connection, we shall present the following result [8, theorem 10.1]

Theorem 2.3. *If $\gamma_n\colon J^n \to M$, J^n being neighborhood in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \le \dim M$, is a C^1 injective mapping, then a necessary and sufficient condition for the existence of frame(s) normal over $\gamma_n(J^n)$ for some linear connection in a vector bundle (E, π, M) is, in some neighborhood (in $\mathbb{R}^n$) of every $s \in J^n$, their (3-index) coefficients to satisfy the equations*

$$\bigl(R_{\mu\nu}(-\Gamma_1 \circ \gamma_n, \dots, -\Gamma_{\dim M} \circ \gamma_n)\bigr)(s) = 0, \qquad \mu, \nu = 1, \dots, n \tag{2.31}$$

where $R_{\mu\nu}$ (in a coordinate frame $\{E_\mu = \frac{\partial}{\partial x^\mu}\}$ in a neighborhood of $\gamma_n(s) \in M$) are given via

$$\begin{aligned} &R_{\mu\nu}(-\Gamma_1 \circ \gamma_n, \dots, -\Gamma_{\dim M} \circ \gamma_n) \\ &\quad := -\frac{\partial \Gamma_\mu(\gamma_n(s))}{\partial s^\nu} + \frac{\partial \Gamma_\nu(\gamma_n(s))}{\partial s^\mu} + \Gamma_\mu(\gamma_n(s))\Gamma_\nu(\gamma_n(s)) - \Gamma_\nu(\gamma_n(s))\Gamma_\mu(\gamma_n(s)). \end{aligned} \tag{2.32}$$

for $\mu, \nu = 1, \dots, n$. Here $\{s^1, \dots, s^n\}$ are Cartesian coordinates in $\mathbb{R}^n$ and the local coordinates $\{x^\mu\}$ on M are such that $x(\gamma_n(s)) = (s, \boldsymbol{t}_0)$ for some fixed $\boldsymbol{t}_0 \in \mathbb{R}^{\dim M - n}$ and in a neighborhood of $\gamma_n(s)$ in M the coordinates of a point in it are $(s', \boldsymbol{t})$ for some $s' \in J^n$ and $\boldsymbol{t} \in \mathbb{R}^{\dim M - n}$

For details concerning the construction of the local coordinates $\{x^\mu\}$ in theorem 2.3, the reader is referred to [11, 8]

From (2.31) an immediate observation follows (see [11, sect. 5]): strong normal frames always exist at every single point ($n = 0$) or/and along every C^1 injective path ($n = 1$). Besides, these are the *only cases* when normal frames *always exist* because for them (2.31) is identically valid. On submanifolds with dimension greater than or equal to two normal frames exist only as an exception if (and only if) (2.31) holds. For $n = \dim M$ equations (2.31) express the flatness of the corresponding linear connection.

If on U exists a frame $\{e_i\}$ normal for ∇, then all frames $\{e'_i = A^j_i e_j\}$ which are normal over U can easily be described: for the normal frames, the matrix $A = [A^j_i]$ must be such that $E_\mu(A)|_U = 0$ for some (every) frame $\{E_\mu\}$ over U in $T(M)$.

3. Electromagnetic potentials

Recall [19, 20], classical electromagnetic field is described via a real 1-form A over a 4-dimensional real manifold M (endowed with a Riemannian metric g and) representing the spacetime model and, usually, identified with the Minkowski space M^4 of special relativity

or the Riemannian space V_4 of general relativity.[5] The electromagnetic field itself is represented by the two-form $F = \mathrm{d}A$, where "d" denotes the exterior derivative operator, with local components (in some local coordinates $\{x^\mu\}$)

$$F_{\mu\nu} = -\frac{\partial A_\mu}{\partial x^\nu} + \frac{\partial A_\nu}{\partial x^\mu}. \tag{3.1}$$

As is well known, the electromagnetic field, the Maxwell equations describing it, and its (minimal) interactions with other objects are invariant under a gauge transformation

$$A_\mu \mapsto A'_\mu = A_\mu + \frac{\partial \lambda}{\partial x^\mu} \tag{3.2}$$

or $A \mapsto A' = A + \mathrm{d}\lambda$, where λ is a C^2 function. As is almost evident, the electromagnetic field is invariant under simultaneous changes of the local coordinate frame, $E_\mu = \frac{\partial}{\partial x^\mu} \mapsto E'_\mu = B^\nu_\mu E_\nu$ with $B^\nu_\mu := \frac{\partial x^\nu}{\partial x'^\mu}$, and a gauge transformation (3.2):

$$A_\mu \mapsto A'_\mu = B^\nu_\mu A_\nu + E'_\mu(\lambda) = B^\nu_\mu \Big(A_\nu + \frac{\partial \lambda}{\partial x^\nu} \Big). \tag{3.3}$$

A simple calculation shows that under the transformation (3.3), the quantities (3.1) transform like components of an (antisymmetric) tensor,

$$F_{\mu\nu} \mapsto F'_{\mu\nu} = B^\sigma_\mu B^\tau_\nu F_{\sigma\tau} \tag{3.4}$$

due to which the 2-form F remains unchanged, $F = \mathrm{d}A = \mathrm{d}A'$. Notice, above A'_μ are *not* the components of A in $\{E'_\mu\}$ unless $\lambda = \mathrm{const}$ while $F'_{\mu\nu}$ *are* the components of F in $\{E'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \mathrm{d}x^\nu\}$.

Comparing (3.3) with (2.5), we see that the former equation is a special case of the latter one if we put in it $\dim \pi^{-1}(x) = 1$, $x \in M$, $\Gamma_\mu = A_\mu$ and $A = \lambda$. That simple observation reflects a known fundamental result [21,17,20]: *from geometrical view-point the electromagnetic potentials are coefficients of a covariant derivative (linear connection) (in a given fields of bases* – vide infra*) in one-dimensional real vector bundle over the spacetime.* Precisely, let (E, π, M) be one-dimensional real vector bundle over the spacetime manifold M and ∇ be a linear connection in it.[6] If $\{e\}$ is a 1-vector frame in E over $U \subseteq M$ [7] and $\{E_\mu\}$ is a frame in the tangent bundle to M over U, then the coefficients $\Gamma_\mu (\equiv \Gamma^1{}_{1\mu})$ of ∇ in $(\{e\}, \{E_\mu\})$ are defined by (see (2.3))

$$\nabla_{E_\mu} e = \Gamma_\mu e \tag{3.5}$$

and a non-degenerate change

$$(\{e\}, \{E_\mu\}) \mapsto (\{e' = \lambda e\}, \{E'_\mu = B^\nu_\mu E_\nu\}) \tag{3.6}$$

for $\lambda, B^\nu_\mu \colon U \to \mathbb{R}$, with λ being of class C^1 and $\det[B^\nu_\mu] \neq 0$, entails (see (2.4))

$$\Gamma_\mu \mapsto \Gamma'_\mu = B^\nu_\mu (\Gamma_\nu + E_\nu(\lambda)). \tag{3.7}$$

Conversely, any geometrical object with components Γ_μ in $(\{e\}, \{E_\mu\})$ with transformation law (3.7) defines a unique linear connection ∇ with coefficients (coefficients' matrices) Γ_μ via (3.5). If we now specify $\{E_\mu\}$ as a coordinate frame induced by local coordinates $\{x^\mu\}$,

[5] The particular choice of M is insignificant for the following.

[6] A one-dimensional vector bundle is called *line bundle*.

[7] We suppress the index 1 related to the frames in E, i.e. we write e for e_1. However, if $u(x) \in \pi^{-1}(x)$, $x \in U$, we have to write e.g. $u(x) = u^1(x) e(x)$ to distinguish $u(x) \in \pi^{-1}(x)$ from $u^1(x) \in \mathbb{R}$.

i.e. $E_\mu = \frac{\partial}{\partial x^\mu}$, we see that (3.7) and (3.3) are identical up to the identification $\Gamma_\mu = A_\mu$, which completes the proof of our assertion.

A new moment in the geometrical treatment of the electromagnetic potentials as coefficients of a linear connection is the clear meaning of the gauge transformations (3.2) as transformation of the potentials under the change

$$(\{e\}, \{E_\mu\}) \mapsto (\{e' = \lambda e\}, \{E_\}) \tag{3.8}$$

corresponding only to a rescaling with factor $\lambda\colon U \to \mathbb{R}\setminus\{0\}$ of the single frame vector field e of the vector bundle frame $\{e\}$ in $\pi^{-1}(U) \subseteq E$ over a set $U \subseteq M$.

In this context, the different gauge conditions, which are frequently used, find a natural interpretation as a partial fix of the class of frames in the bundle space employed. For instance, any one of the gauges in the table on this page corresponds to a class of frames for which (3.3) holds for $B^\nu_\mu = \delta^\nu_\mu$, δ^ν_μ being the Kroneker deltas, and λ subjected to a condition given in the table.[8]

Gauge	Condition on A	Condition on λ	Condition on φ
Lorentz	$\partial^\mu A_\mu = 0$	$\partial^\mu\partial_\mu\lambda = 0$	$\partial^\mu\partial_\mu\varphi = -\partial^\mu\partial_\mu\lambda$
Coulomb[a]	$\partial^k A_k = 0$	$\partial^k\partial_k\lambda = 0$	$\partial^k\partial_k\varphi = -\partial^k\partial_k\lambda$
Hamilton	$A_0 = 0$	$\lambda(x) = \lambda(x^1, x^2, x^3)$	$\varphi(x) = \varphi(x^1, x^2, x^3)$
Axial	$A_3 = 0$	$\lambda(x) = \lambda(x^0, x^1, x^2,)$	$\varphi(x) = \varphi(x^0, x^1, x^2)$

[a]In this raw the summation over k is from 1 to 3.

In the table on the current page φ is a C^1 function describing the arbitrariness in the choice of λ, i.e. if a gauge condition is valid for λ, then it holds also for $\lambda + \varphi$ instead of λ.

If an electromagnetic field exists on an open set $U \subseteq M$, then its potentials admit an equivalent geometrical interpretation as 3-index coefficients of a linear transport along paths which is, in fact, the parallel transport along paths for the linear connection whose coefficients coincide with the field's potentials (see corollaries 2.2 and 2.3).

If one considers a free (pure) electromagnetic field, the bundle space of the line bundle on which the field can be described as a linear connection, remains undetermined.

Suppose now an electromagnetic field exists on some submanifold N of M and $\dim N < \dim M$. (With some approximation such fields can be realized.) In this case one cannot interpret the electromagnetic potentials as coefficients of a linear connection on a line bundle (E, π, M) over the spacetime M. But such an interpretation is possible on the restricted subbundle $(E, \pi, M)|_N = (\pi^{-1}(N), \pi|_{\pi^{-1}(N)}, N)$ for which one can repeat *mutatis mutandis* the above considerations. However the interpretation of field potentials as 3-index coefficients of a linear transport along paths can be retained. To demonstrate that, consider one-dimensional bundle (E, π, M) and a linear transport L along paths in it such that the matrices of its coefficients satisfy the condition

$$\mathbf{\Gamma}(s;\gamma) = \Gamma_\mu(\gamma(s))\dot\gamma^\mu(s) \qquad \text{for any } C^1 \text{ path } \gamma\colon J \to N \text{ and } s \in J \tag{3.9}$$

for some matrix-valued functions Γ_μ on N in any frame $\{e\}$ in E over N and frame $\{E_\mu\}$ in the tangent bundle space over N. Otherwise the transport L is completely arbitrary, e.g. we can require

$$\mathbf{\Gamma}(t;\beta) = \tilde\Gamma_\mu(\beta(t))\dot\beta^\mu(t) \qquad \text{for any } C^1 \text{ path } \beta\colon \tilde J \to M \text{ and } t \in \tilde J \tag{3.10}$$

for matrix-valued functions $\tilde\Gamma_\mu$ on M such that

$$\tilde\Gamma_\mu(x) = \Gamma_\mu(x) \qquad \text{for } x \in N. \tag{3.11}$$

[8]Below M is supposed to be endowed with a Riemannian metric $g_{\mu\nu}$, the coordinates to be numbered as x^0, x^1, x^2, and x^3, x^0 to be the 'time' coordinate, $\partial_\mu := \partial/\partial x^\mu$, and $\partial^\mu := g^{\mu\nu}\partial_\nu$ with $[g^{\mu\nu}] := [g_{\mu\nu}]^{-1}$.

If we identify $\Gamma_\mu(x)$, $x \in N$, with the electromagnetic potentials A_μ, then A_μ are 3-index coefficients of any linear transport L along paths for which equation (3.9) holds. In invariant terms, equation (3.9) can be rewritten as

$$L^\gamma = \mathsf{P}^\gamma \qquad \text{for } \gamma\colon J \to N \tag{3.12}$$

where P is the parallel transport along paths in $(\pi^{-1}(N), \pi|_{\pi^{-1}(N)}, N)$ whose coefficients are the electromagnetic potentials (on N). Obviously, the condition (3.12) does not define L uniquely for paths which do not lie entirely in N.

4. Equivalence principle in gravitation

The primary role of the principle of equivalence is to ensure the transition from general to special relativity. It has quite a number of versions, known as weak and strong equivalence principles [22, pp. 72–75], any one of which has different, sometimes non-equivalent, formulations. In the present paper only the strong(est) equivalence principle is considered. Some of its formulations can be found in [7].

Freely speaking, an inertial frame for a physical system is a one in which the system behaves in some aspects like a free one, i.e. such a frame 'imitates' the absence (vanishment) of some forces acting on the system. Generally inertial frames exist only locally, e.g. along injective paths, and their existence does not mean the vanishment of the field responsible for a particular force. The best known examples of this kind of frames are for the gravitational field. Below we rigorously generalize these ideas to classical electrodynamics.

In [7] it was demonstrated that, when gravitational fields are concerned, the inertial frames for them are the normal ones for the linear connection describing the field and they coincide with the (inertial) frames in which the special theory of relativity is valid. The last assertion is the contents of the (strong) equivalence principle. In the present section, relying on the ideas at the end of [7, sec. 5], we intend to transfer these conclusions to the area of classical electrodynamics.

The normal frames are the mathematical concept corresponding to/describing the physical one of inertial frames (of reference). However, as we have seen in Sect. 2, frames normal for a linear connection always exist at a given single point and/or along (injective) path and on more general sets they exist only in some exceptional case (see, e.g., theorem 2.3). This means that the (strong) equivalence principle is valid at a given single point or path and on submanifolds of the spacetime of dimension greater or equal to two it may be true only for some special gravitational fields; in particular, on open sets (which are submanifolds of dimension $\dim M = 4$) it holds iff the linear connection, describing the field, is curvature free, which physically is interpreted as absence of the gravity field.

The above conclusions have a general validity and concern non only the general relativity but rather any gravitational theory in which the gravitational field strength is identified with the coefficients of a linear connection (in the tangent bundle over the spacetime).

5. Equivalence principle in electrodynamics

Consider a one-dimensional vector bundle (E, π, M) over the spacetime M in which a classical electromagnetic field is described via a linear connection ∇ (or parallel transport P) whose (3-index) coefficients coincide with the field's potentials A_μ in a pair $(\{e\}, \{E_\mu\})$ of frames $\{e\}$ in E and $\{E_\mu\}$ in the tangent bundle space to M.

A frame of reference will be called *inertial* for an electromagnetic field on a set $U \subseteq M$ if in it the field strength vanishes on U, i.e. if in it we have

$$A'_\mu|_U = 0 \tag{5.1}$$

as the field strength is (locally) identified with the electromagnetic potentials. If an inertial frame exists on U, in it an electrically charge particle (body) will behave in U like a neutral one (if it is entirely situated in U). Obviously, the mathematical object corresponding to an inertial frame of reference on U for an electromagnetic field is a frame (in E) normal on U for the linear connection describing it. The simple observation of the *coincidence of inertial and normal frames is the contents of the equivalence principle in classical electrodynamics.*[9] We can equivalently restate it as the assertion of coincidence of the inertial frames and strong normal frames for the parallel transport along paths describing the field (which is generated by or generates the linear connection corresponding to the field).

Comparing (3.1) with (2.32), we get[10]

$$F_{\mu\nu} = R_{\mu\nu}(-A_0, -A_1, -A_2, -A_3). \tag{5.2}$$

Thus, the electromagnetic field tensor F is completely responsible for the existence of frames normal for the parallel transport P (theorem 2.3). For example, if U is an open set, frames normal on $U \subseteq M$ for P exist iff $F|_U = 0$, i.e. if electromagnetic field is missing on U.[11] Also, if N is a submanifold of M, frames normal on U for P exist iff in the special coordinates $\{x^\mu\}$, described in theorem 2.3, is valid $F_{\alpha\beta}|_U = 0$ for $\alpha, \beta = 1, \dots, \dim N$. In the context of [12, theorem 4.1], we can say that an electromagnetic field admits frames normal on $U \subseteq M$ iff the corresponding to it linear transport P is path-independent on U (along paths lying entirely in U). Thus, if P is path-dependent on U, the field does not admit frames normal on U. This important result is the classical analogue of the quantum effect, know as the Aharonov-Bohm effect [24, 25], whose essence is that the electromagnetic potentials directly, not only through the field tensor F, can give rise to observable physical results.

Let us now turn our attention to the physical meaning of the normal frames corresponding to a given electromagnetic field which is described, as pointed above, via a parallel transport P along paths in 1-dimensional vector bundle over the space-time M.

Suppose P is Euclidean on a neighborhood $U \subseteq M$. As a consequence of (5.2) and [12, theorem 5.1], we have $F|_U = \mathrm{d}A|_U = 0$, i.e. on U the electromagnetic field strength vanishes and hence the field is a pure gauge on U,

$$A_\mu|_U = \frac{\partial f_0}{\partial x^\mu}\Big|_U \tag{5.3}$$

for some C^1 function f_0 defined on an open neighborhood containing U or equal to it. In a frame $\{e'\}$ normal on U for P vanish the 2-index coefficients of P along any path γ in U:

$$\mathbf{\Gamma}'(s;\gamma) = A'_\mu(\gamma(s))\dot\gamma^\mu(s) = 0 \tag{5.4}$$

for every $\gamma\colon J \to U$ and $s \in J$. Using (5.3), it is trivial to see that any transformation (3.7) with

$$\lambda = -f_0 \tag{5.5}$$

transforms A_μ into A'_μ such that

$$A'_\mu|_U = 0 \tag{5.6}$$

[9] Generally a frame of reference is a more complex concept than a coordinate system or a field of bases in (some) bundle space of a vector bundle. However, the other characteristics and properties of the physical concept of a reference frame are inessential in the context of the present investigation.

[10] Below we assume the Greek indices to run over the range 0, 1, 2, 3.

[11] Elsewhere we shall prove that the components $F_{\mu\nu}$ completely describe the curvature of P which agrees with the interpretation of $F_{\mu\nu}$ as components of the curvature of a connection on a vector bundle in the gauge theories [21, 20, 23]. The general situation is similar: the quantities (2.32) determine the curvature of a transport with coefficients' matrix (2.28).

(irrespectively of the frames $\{E_\mu\}$ and $\{E'_\mu\}$ in the tangent bundle over M). Hence, by (5.4) the one-vector frame $\{e' = \mathrm{e}^{-f_0} e\}$ in the bundle space E is normal for P on U. Therefore in the frame $\{e'\}$, there vanish not only the 2-index coefficients of P but also its *3-index* ones, i.e. $\{e'\}$ is a frame strong normal on U for P. Applying (3.3) one can verify, *all frames strong normal on a neighborhood U for P are obtainable from $\{e'\}$ by multiplying its vector e' by a function f such that $\frac{\partial f}{\partial x^\mu}\big|_U = 0$*, i.e. they are $\{b\mathrm{e}^{-f_0} e\}$ with $b \in \mathbb{R}\backslash\{0\}$ as U is a neighborhood. Thus, every frame normal on a neighborhood U for P is strong normal on U for P and vice versa.

So, in a frame inertial on $U \subseteq M$ for an electromagnetic field it is not only a pure gauge, but in such a frame its potentials vanish on U. Relying on the results obtained (see also [9, 10, 11]), we can assert the existence of frames inertial at a single point and/or along paths without self-intersections for every electromagnetic field, while on submanifolds of dimension not less than two such frames exist only as an exception if (and only if) some additional conditions are satisfied, i.e. for some particular types of electromagnetic fields.

6. Conclusion

In this paper we introduced normal an inertial frames for classical electromagnetic field. The coincidence of these two types of frames expresses the equivalence principle for that field. Generally this principle is a provable theorem and it is always valid at any single point of along given path (without selfintersections) as these are the only case when normal frames for a linear connection always exist.

The considerations of the equivalence principle in electrodynamics were base on the interpretation of a classical electromagnetic field as a linear connection or (the generating it or generated by it) parallel transport along paths in one-dimensional vector bundle over the spacetime. Within the electrodynamics, i.e. for a free electromagnetic field, that bundle remains unspecified. However, if an interaction of electromagnetic field and some other field is investigated, the bundle mentioned can be identified or uniquely connected with a bundle (over the spacetime) whose sections represent the latter field. Moreover, in such a situation the equivalence principle can be used to justify the so-called minimal coupling (principle), i.e. the description of the interaction of some field with an electromagnetic one via a replacement of the ordinary partial derivatives in the free Lagrangian of the former field with covariant ones relative to the linear connection representing the latter field.

At last, we would like to mention that the existence of a normal/inertial frames on some subset does not generally imply vanishment of the field on this set if it is not an open set.

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